Def: Let X, Y be rvs. The covariance of X and Y, write $\operatorname{cov}(X,Y)$, is defined by

$$\operatorname{cov}(X,Y) \equiv E[(X-\mu_X)(Y-\mu_Y)]$$

Prop: If X, Y are independent, then $\operatorname{cov}(X,Y)=0$

Prop: Let $X_1, X_2 \ldots X_N$ $(1 \le N < \infty)$ be rvs. Then

$$\operatorname{var}(X_1+\cdots+X_N) = \sum_{i=1}^{N} \operatorname{var}(X_i) + \sum_{i \neq j} \operatorname{cov}(X_i, X_j)$$

In particular, if X_1, X_N are indep, then

$$\operatorname{var}(X_1+\cdots+X_N) = \operatorname{var}(X_1)+\cdots+\operatorname{var}(X_N)$$

Pf: $\operatorname{var}(X_1+\cdots+X_N) = E[(X_1+\cdots+X_N - \mu_{X_1+\cdots+X_N})^2]$

$$= E\left(\left[\sum_{i=1}^{N}(X_i-\mu_{X_i})\right]^2\right)$$

$$= \sum_{i=1}^{N} E(X_i-\mu_{X_i})^2 + \sum_{i\neq j} E[(X_i-\mu_{X_i})(X_j-\mu_{X_j})]$$

$\square$

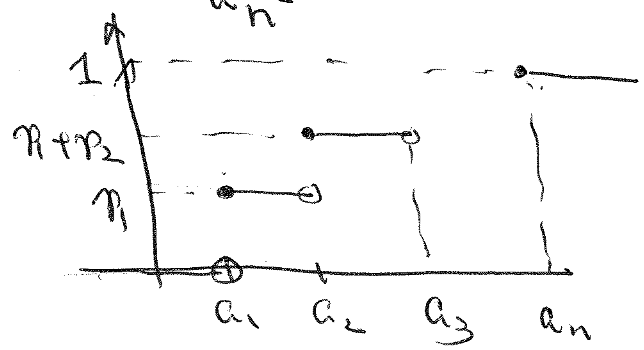
lemma: Let $a_1 < \dots < a_n$ and $(p_1, \dots, p_n) \in \mathbb{R}^n$
with $0 \le p_i \le 1$, $\sum_{i=1}^{n} p_i = 1$

Let $f: \{a_1, \dots, a_n\} \to \{p_1, \dots, p_n\}$ be a function
such that $f(a_i) = p_i$.

Then there is a prob space (Ω, ξ, P)
and a rv $X: \Omega \to \{a_1, \dots, a_n\}$ such that
~~f(a_i)~~ $P\{X = a_i\} = p_i$ for all $i = 1, \dots, n$.

pf: Define $F: \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \begin{cases} 0 & \cancel{t \le a_1}\ t < a_1 \\ p_1 + \dots + p_i & a_i \le t < a_{i+1} \\ 1 & a_n \le t \end{cases}$$



Then $\exists\ (\Omega, \xi, P)$ and a rv $X: \mathbb{R} \to \mathbb{R}$
st $F_X = F$ ie:

$$F_X(t) = F(t), \quad \forall\ t \in \mathbb{R}$$

Then $P\{X = a_i\} = \cancel{P\{X \le a_i\}} \lim_{n\to\infty} P\{a_i - \tfrac{1}{n} < X \le a_i\}$
$= \lim_{n\to\infty} [F_X(a_i) - F_X(a_i - \tfrac{1}{n})] = p_i$

Def: Assume that there are n independent trials and the probability of success in each trial is p. Let X denote the number of successes. Then X is called a binomial random variable with parameter n, p. Write $X \sim b(n,p)$

Remark: There is a prob sp (Ω, Σ, P) and a rv X s.t. $X \sim b(n,p)$

Define: $\Omega = \{(x_1, .., x_n) \mid x_i = 0 \text{ or } 1\}$

$\Sigma = \mathcal{P}(\Omega)$

Define $P\{(x_1, .., x_n)\} = p^i (1-p)^{n-i}$

~~where i = the number of x_k s.t. $x_k = 1$~~

and $X : \Omega \to \{0, 1, .., n\}$

$X(x_1, .., x_n) = i$

where i = the number of x_k s.t. $x_k = 1$.

Then $X \sim b(n,p)$

Prop: Let $X \sim b(n,p)$. Then

(i): $P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le k \le n$

(ii) $E(X) = np$

(iii) $\mathrm{Var}(X) = np(1-p)$.

pf: (ii). $E(X) = \sum_{k=0}^{n} k P\{X=k\}$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} p \cdot p^{k-1} (1-p)^{n-1-(k-1)}$$

$$= np(p+(1-p))^{n-1} = np.$$

(iii). $\mathrm{var}(X) = E(X^2) - E(X)^2$

$$E(X^2) = \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k\, n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{(k-1)\, n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} + \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} + \sum_{j=0}^{n-1} \frac{n!}{j!(n-1-j)!} p^{j+1} (1-p)^{n-1-j}$$

$(i=k-2) \qquad (j=k-1)$

$$= \sum_{i=0}^{n-2} \frac{n!}{i!(n-2-i)!} p^{i+2} (1-p)^{n-2-i} + \text{″}$$

$= p^2 n(n-1)[p+(1-p)]^{n-2} + np(p+(1-p))^{n-1}$ (4)

$\therefore \ Var(X) = E(X^2) - E(X)^2$

$= n(n-1)p^2 + np - n^2p^2$

$= -np^2 + np = np(1-p)$ $\square$

Alternative proof:

Define $X_i : \Omega \longrightarrow \mathbb{R}$ by

$$X_i = \begin{cases} 1 & \text{if } i\text{-th trial success} \\ 0 & \text{otherwise} \end{cases}$$

Then $X = X_1 + \cdots + X_n$

Note: $E(X_i) = p$

$var(X_i) = E(X_i^2) - E(X_i)^2$

$= E(X_i) - E(X_i)^2$

$= p - p^2 \quad \forall i$

$\square$

Observation:

Let $X_n \sim b(n,p)$. and $\lambda = np$.

Fix λ. and $k \in \mathbb{N}$.

$$P\{X_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= 1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\cdot\frac{1}{k!}\lambda^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\longrightarrow \frac{1}{k!}\lambda^k e^{-\lambda}$$

Def: A discrete rv $X: \Omega \to \{0,1,2,\dots\}$ is called a Poisson rv with parameter $\lambda > 0$. if

$$P\{X = k\} = \frac{1}{k!} e^{-\lambda} \cdot \lambda^k, \quad \forall k = 0,1,2\dots$$

Remark: Define: $F: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F(t) = \begin{cases} 0 & t < 0 \\ \cancel{\frac{1}{k!} e^{-\lambda} \lambda^k} & \cancel{k \le t < k+1} \\ \sum_{i=0}^{k} \frac{1}{i!} e^{-\lambda} \cdot \lambda^i & k \le t < k+1 \end{cases}$$

Note F is right cont and (43)

$$\lim_{t\to\infty} F(t) = \lim_{k\to\infty} \sum_{i=0}^{k} e^{-\lambda}\frac{\lambda^i}{i!} = 1$$

$\therefore$ Such a Poisson rv exists ~~for all~~. $\square$

Prop: If X is a Poisson rv with a parameter $\lambda > 0$,

then $E(X) = \lambda = var(X)$

pf: $\vdash_1$: $E(X) = \lambda$

Note: $E(X) = \sum_{k=0}^{\infty} k P\{X=k\} = \sum_{k=0}^{\infty} k \cdot \frac{1}{k!} \lambda^k e^{-\lambda}$

$$= \left(\sum_{k=1}^{\infty} \frac{1}{(k-1)!} \lambda^{k}\right) e^{-\lambda}$$

$$= \left(\sum_{j=0}^{\infty} \frac{1}{j!} \lambda^{j+1}\right) e^{-\lambda} = \lambda$$

$(j = k-1)$

$\vdash_2$: $var(X) = \lambda$: Note $var(X) = E(X^2) - E(X)^2$

Note: $E(X^2) = \sum_{k=0}^{\infty} k^2 P\{X=k\} = \sum_{k=1}^{\infty} k^2 e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$$= \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k (k-1)}{(k-2)!} + \left(\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}\right) e^{-\lambda}$$

$$= e^{-\lambda} \cdot \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} + \left(\sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!}\right) e^{-\lambda}$$

$j = k-1$

$$= e^{-\lambda}\left(\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}\right)\lambda^2 + \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}\right)\lambda e^{-\lambda}$$

$$= \lambda^2 + \lambda \qquad \therefore var(X) = \lambda \qquad \square$$